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An Approach for Reducing Computational Requirements in Modal Identification

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Sparse Time Domain Algorithm

LIKE other time domain modal identification techniques, the sparse time domain algorithm (STD) is based on using the free decay or impulse time response functions of the structure under test as the data for identification. Assuming linearity, or series expansion for a nonlinear system,[1] lumped parameter model representation of distributed parameter systems,[2] equivalent viscous damping and allowing for measurement noise, these responses are expressed as functions of the complex vectors ψ and the characteristic roots λ from which all modal parameters can be determined. If the initial condition's constants are implicitly contained in ψ, these responses are

$$\{x(t)\} = \sum_{i=1}^{2n} \{\psi_i\} e^{\lambda_i t} + \{n(t)\} \tag{1}$$

To reduce the effects of measurement noise on the identification accuracy, if these responses are sought to contain n structural modes, an oversized identification model of m degrees of freedom is used. In general, m is greater than n to an extent dependent on the noise-to-signal ratio of the measurements.

If the measurements are sampled at a sampling frequency of f_s Hz and the time between samples is Δt where

$$f_s = 1/\Delta t \tag{2}$$

a response matrix $[\phi]$, $(r \times 2m)$ where $r > 2m$, is constructed such that

$$\phi_{ij} = x_i\{k + (j-1)\ell\Delta t\} \tag{3}$$

$$i = 1,2,\ldots r \text{ and } j = 1,2,\ldots 2m$$

where k is an arbitrary constant integer and ℓ a selected integer that is determined according to the antialiasing condition ($1/\ell\Delta t$ greater than twice the maximum frequency in the response). In matrix form, Eqs. (1) and (3) can be written as

$$[\phi] = [\psi]\ [\Lambda] \tag{4}$$

where

$$\Lambda_{ij} = e^{\lambda_i\{k + (j-1)\ell\Delta t\}}, \quad (i,j = 1,2,\ldots 2m)$$

In Eqs. (3) or (4), the measurement x_i may be 1) the actual measurement on a test structure, 2) the pseudomeasurement that is an actual measurement delayed in time, 3) the actual or pseudomeasurement from a different test with different initial conditions (or initial excitation location), or 4) the actual or pseudomeasurements from a multiple excitation test.

To convert Eq. (4) to the desired eigenvalue problem, a similar response matrix $[\hat{\phi}]$ is constructed such that

$$\hat{\phi}_{ij} = \phi_{i,j+1} \tag{5}$$

and, by simple matrix algebra, it can be shown that

$$[\phi][\Lambda]^{-1}[\alpha] = [\hat{\phi}][\Lambda]^{-1} \tag{6}$$

$$[H][\Lambda]^{-1} = [\Lambda]^{-1}[\alpha] \tag{7}$$

where $[\alpha]$ is a diagonal matrix whose elements are

$$\alpha_i = e^{\lambda_i \ell \Delta t}$$

and H satisfies the equation

$$[\phi][H] = [\hat{\phi}]$$

Because of the relation between $[\phi]$ and $[\hat{\phi}]$, the $[H]$ matrix takes the form

$$\begin{bmatrix} 0 & 0 & 0 & 0\ldots 0 & a_1 \\ 1 & 0 & 0 & 0\ldots 0 & a_2 \\ 0 & 1 & 0 & 0\ldots 0 & a_3 \\ 0 & 0 & 1 & 0\ldots 0 & a_4 \\ \multicolumn{5}{c}{\text{----------}} \\ & & & & \\ \multicolumn{5}{c}{\text{----------}} \\ 0 & 0 & 0 & 0\ldots 1 & a_{2m} \end{bmatrix} \tag{8}$$

which is a sparse upper Hessenberg matrix with only one column $\{a\}$ of information. To compute such a vector, the equation

$$[\phi]\{a\} = \{\hat{\phi}_{2m}\} \tag{9}$$

is used and, since Eq. (9) is an overdetermined system of equations, the least squares solution is used to give

$$[\phi]^T[\phi]\{a\} = [\phi]^T\{\hat{\phi}_{2m}\} \tag{10}$$

or

$$[B]\{a\} = \{b\} \tag{11}$$

and since $[B] = [\phi^T\phi]$, it is symmetrical and positive definite and Eq. (11) can be solved by using the stable Cholesky decomposition[3] to solve for vector $\{a\}$.

To reduce any bias errors associated with solving for $\{a\}$ that may arise because of the inherent statistically biased errors of the least squares solution, a double least squares solution may be performed. Such a step seems unnecessary since, as it will be shown later, the classical least squares ap-

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proach in this case yields good identification accuracy with no biased errors.

After computing the vector $\{a\}$ from Eq. 11, the $[H]$ matrix is composed as shown in Eq. (8). Then, the *QR* algorithm is directly used, without need for transformation, to compute the eigenvalues α. The eigenvectors of $[H]$, which are the columns of $[\Lambda]^{-1}$, may or may not be computed, depending on the approach used to calculate the modal vectors ψ, as will be discussed later. From α, the characteristic roots λ may be calculated from the relation

$$\alpha_i = e^{\lambda_i \ell \Delta t} \tag{12}$$

Thus far, these α correspond to structural as well as computational modes since an oversized identification model is used. At this stage, it will be highly desirable to develop some criteria to separate the physical λ from the computational ones, but at present there is no available approach that can be used for such differentiation.

Computation of Mode Shapes

Two approaches may be used to compute the complex modal vectors ψ. The first approach utilizes Eq. (4) and the readily available $[\Lambda]^{-1}$, which is computed in the eigensolution of $[H]$. Prior to doing so, the matrix $[\Lambda]^{-1}$ can be scaled to obtain the mode shapes with their actual magnitude participation. Such normalization is performed utilizing the simple fact that

$$[\Lambda][\Lambda_s]^{-1} = [I] \tag{13}$$

where the subscript s denotes scaled. To scale the vectors of $[\Lambda]^{-1}$, a complex constant c_i is obtained such that

$$c_i = \sum_{j=1}^{2m} e^{\lambda_i \{k + (j-1)\ell\Delta t\}} \cdot \Lambda_{ji}^{-1} \tag{14}$$

and the scaling is done using the diagonal matrix $[c]$ through the relation

$$[\Lambda_s]^{-1} = [\Lambda]^{-1}[c]^{-1} \tag{15}$$

and, by turn, the modal matrix $[\psi]$ is

$$[\psi] = [\phi][\Lambda_s]^{-1} \tag{16}$$

To save computation, Eq. (16) can be applied only to the measurements of interest, i.e., only certain rows of $[\phi]$.

Since the computation of $[\psi]$ in such a case is straight matrix multiplication, averaging to improve the accuracy is possible by using several sets of overlapping responses to compute several sets of $[\psi]$ that can be averaged.

Table 1 Identification comparison for five modes, 30% *N/S* ratio, 20 deg of freedom identification model

Parameter	Theoretical	ITD (SLS)	ITD (DLS)	STD
f_1, Hz	10.00	10.0171	10.0163	9.9865
f_2	12.00	11.9987	11.9978	11.9862
f_3	15.00	15.0082	14.9869	15.0042
f_4	20.00	20.1145	20.0460	20.0129
f_5	21.00	20.9165	20.9764	21.0278
ζ_1, %	2.00	2.79	1.78	2.11
ζ_2	2.00	3.24	1.90	2.05
ζ_3	2.00	2.89	2.25	2.03
ζ_4	2.00	3.59	1.72	2.02
ζ_5	2.00	3.05	1.63	1.99
Storage, K	—	54.2	57.2	41.4
Execution time, s	—	55.8	107.2	24.8

The second approach does not require the computation of $[\Lambda]^{-1}$, but rather directly implements Eq. (1) and the computed λ to determine the modal vectors.

A response measurement can be rewritten as

$$x_i(t_j) = \sum_{s=1}^{m} e^{a_s t_j}(A_{is}\cos b_s t_j + B_{is}\sin b_s t_j) \tag{17}$$

where

$$\lambda_s = a_s + ib_s \tag{18}$$

Using the measurements of interest, the matrix equation becomes

$$[X] = [A \;\; B][C] \tag{19}$$

where

$$C_{ij} = e^{a_i t_j}\cos b_i t_j, \quad i = 1,\ldots m \tag{20}$$

$$= e^{a_i t_j}\sin b_i t_j, \quad i = m+1,\ldots 2m \quad (j = 1,\ldots 2r,\ r > m) \tag{21}$$

Equation (19) can be solved for $[A \; B]$, which are the actual modal participation coefficients, using the least squares approach. The computational procedure is simplified by using a closed-form formulas for computing the elements of $[CC^T]$.

The two approaches for computing the mode shapes yield comparable results with approximately the same order of magnitude of computational time and storage.

Separation of Physical and Computational Modes

Considering that the α and ψ are computed from an oversized identification model, this section deals with computing a vector of modal confidence factors (MCF)[4] for every computed mode. The MCF vector can then be used to judge if a specific mode is a structural property or a computational degree of freedom.

To compute the MCF vector, the same exact procedure for computing $[\psi]$, or $[A \; B]$, is repeated with the response matrices $[\phi]$ or $[X]$ containing responses that are delayed $p\Delta t$ in time (p is an integer) to compute the new $[\bar{\psi}]$ or $[\bar{A} \; \bar{B}]$. From $[\psi]$ and $[\bar{\psi}]$ or from $[A \; B]$ and $[\bar{A} \; \bar{B}]$, the MCF vector is computed for the jth mode using the formulas,

$$(\text{MCF})_i = \bar{\psi}_{ijn}/(\psi_{ij}e^{\lambda_j p\delta t}) \tag{22}$$

or

$$(\text{MCF})_i = (\bar{A}_{ij} + i\bar{B}_{ij})/\{(A_{ij} + iB_{ij})e^{\lambda_j p\Delta t}\} \tag{23}$$

These MCF's are complex quantities, ranging from (0.0, 0.0) to (1.0, 0) and their reciprocals are used for ones whose magnitudes are greater than unity.

Sample Identification Results

In order to evaluate the proposed approach and compare it to full matrix time domain approaches, a test case was simulated. The simulated responses of 5 modes at 10 measurement locations were constructed using Eq. (1). The mode shapes were arbitrarily selected to be the first five modes of a simply supported beam measured at equal intervals of 1/11th of the beams's length. (Any five orthogonal vectors would do the job.) The frequencies were arbitrarily assigned as 10, 12, 15, 20, and 21 Hz and all modes had 2% damping. To simulate an actual test, a set of randomly generated numbers having a uniform distribution were added to the responses to produce measurements with a 30% rms

N/S ratio. A sampling frequency of 100 Hz was used, the records of which contained 300 points. An identification model had degrees of freedom with ℓ of Eq. (3) equal to 2 and p of Eqs. (22) and (23) for an MCF calculation equal to 3. The pseudomeasurements were obtained by a 2 sample delay of the original 10 measurements. For the least squares solution, the matrix $[\phi]$ was of dimensional 210×40.

The same responses were analyzed using the full matrix Ibrahim time domain (ITD) with both single and double least squares solutions (SLS and DLS).[5] The theoretical and identified results are listed in Table 1 with information on the required computer storage and execution time. The information on the computations should be regarded in the relative sense for comparison, since they are functions of the computer used.

Conclusions

The proposed time domain approach (STD), which is based on the direct use of sparse upper Hessenberg matrix, promises the following advantages:

1) Reduced computer storage and time.
2) Higher identification accuracy.
3) No biased errors were noticed in the identified damping factors.
4) User-selected parameters are reduced.
5) Actual mode participation is automatically computed.
6) The computation of the modal confidence factors is performed as a postidentification procedure, thus having no effect on the identification procedure.
7) The identification procedure can use any number of measurements even larger than the order of the identification model.
8) Multitest and or multi-initial excitation responses can be used simulataneously.

Further evaluation, studies, and applications of the proposed algorithm are needed to further verify its merits.

Acknowledgments

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Readers' Forum

Errata: "Performance of High-Power Spark Gaps"

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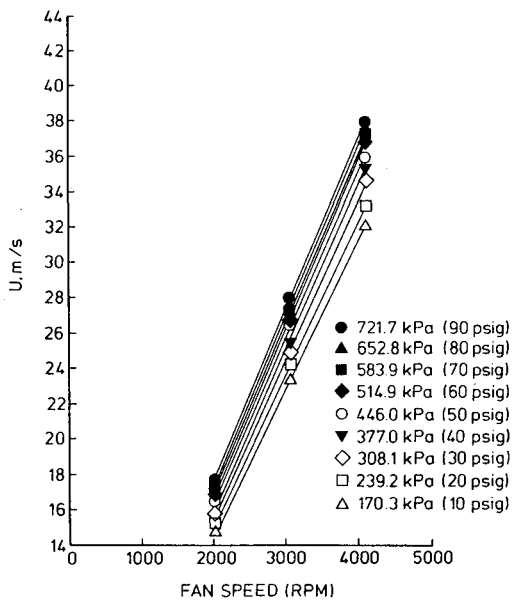
FIGURE 6 on page 1114 was incorrectly printed. The correct figure is shown below.



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